Certificate Checking Static Analyses of Mixed-Precision Floating-Point Arithmetic

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Introduction

Designing floating-point programs is hard

$$\begin{cases} u_0 = 2 \\ u_1 = -4 \\ u_{n+2} = 111 - \dfrac{1130}{u_{n+1}} + \dfrac{3000}{u_{n+1} \cdot u_n} \end{cases}$$

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n	$\widetilde{u_n}$ (128-bits)
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3	9.38
4	7.80
5	7.15
6–29	$\simeq 6$
30	**5.88**
31	3.90
32	-47.97
33	118.52
34–...	$\simeq 100$

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$$\left| u_N - \widetilde{u_N} \right| = \mathbf{94}$$

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$\Longrightarrow$ automatic tools to help developers

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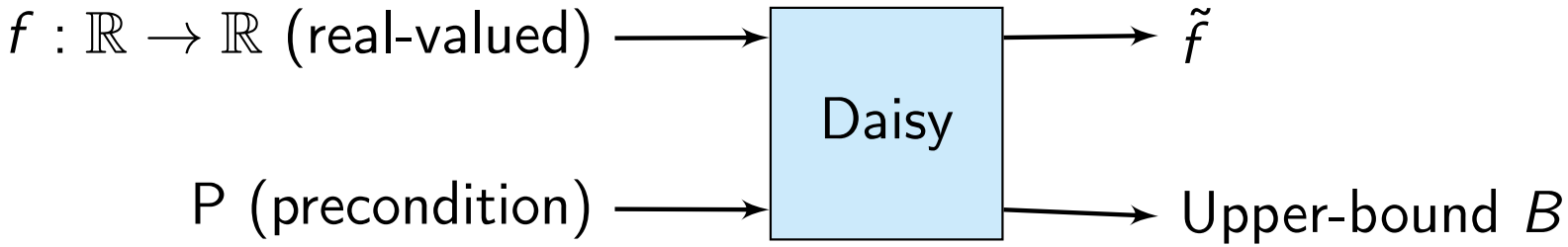
- Upper-bounds the roundoff error $|P(x) - P(\tilde{x})|$
- Soundness property: computes over-approximations of roundoff errors

Daisy, a compiler of real programs



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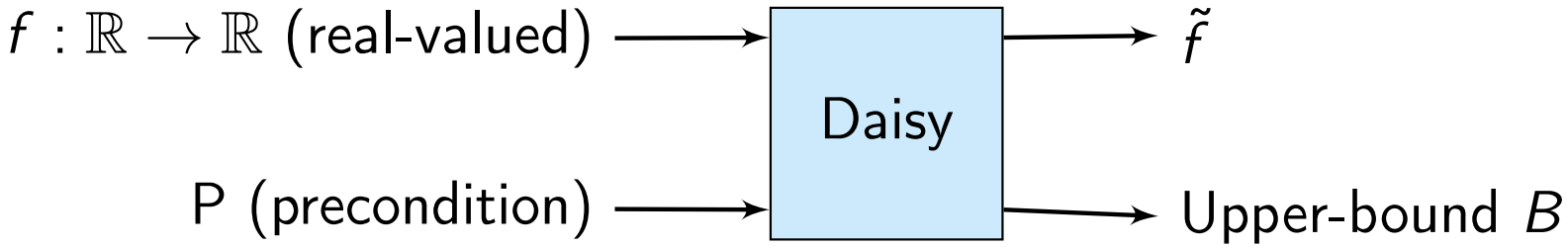

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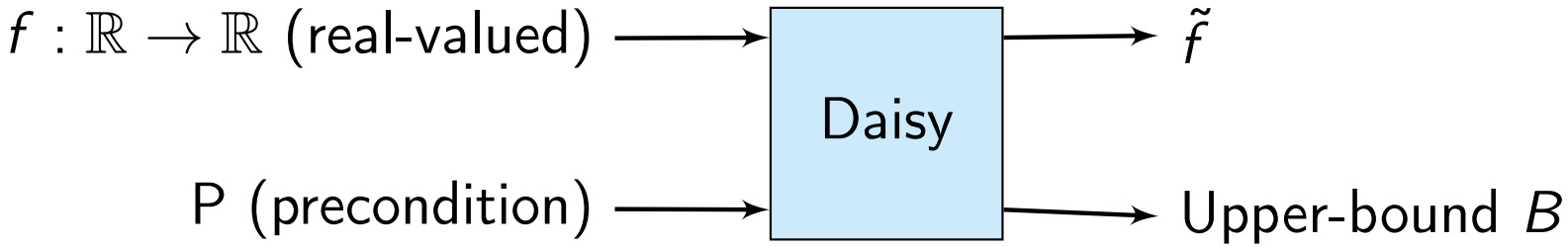

- Soundness: $B \geq \max_{x \in P} \left| f(x) - \tilde{f}(\tilde{x}) \right|$
- Contains a static analysis phase
- Correct optimizations of real-valued programs

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Example: Verasco is a general static analyzer written in Coq.

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I used both Coq and HOL4.

Mixed-precision

```
def sine(a: Double) =
  val t: Double = - a * a * a / 6
  val u: Double = a * a * a * a * a / 120
  a + t + u
```

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$2.51 \cdot 10^{-16}$

Contribution

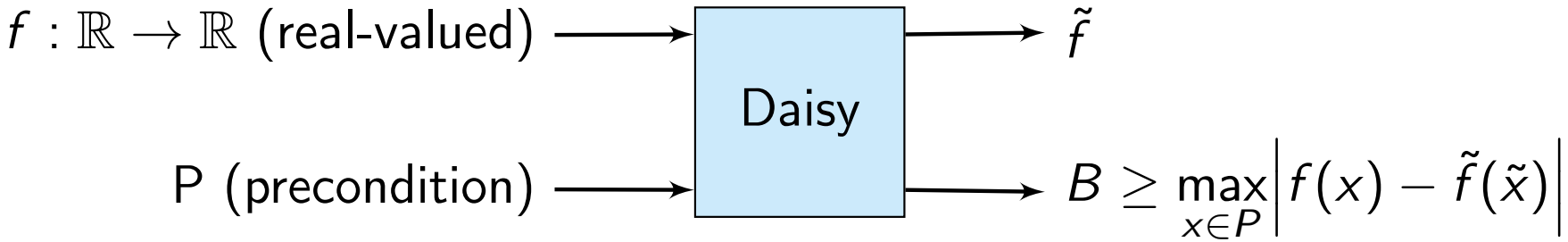
Before, Daisy supported:

- formal validation of uniform-precision analyses
- mixed-precision analyses

Contribution: I extended the formal validation to mixed-precision.

Daisy

Extending Daisy to Mixed-Precision



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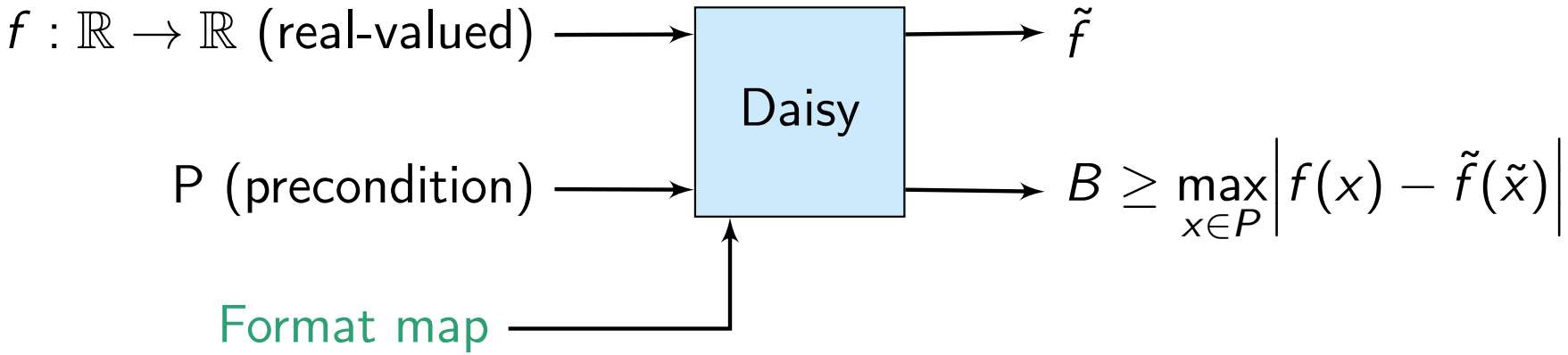

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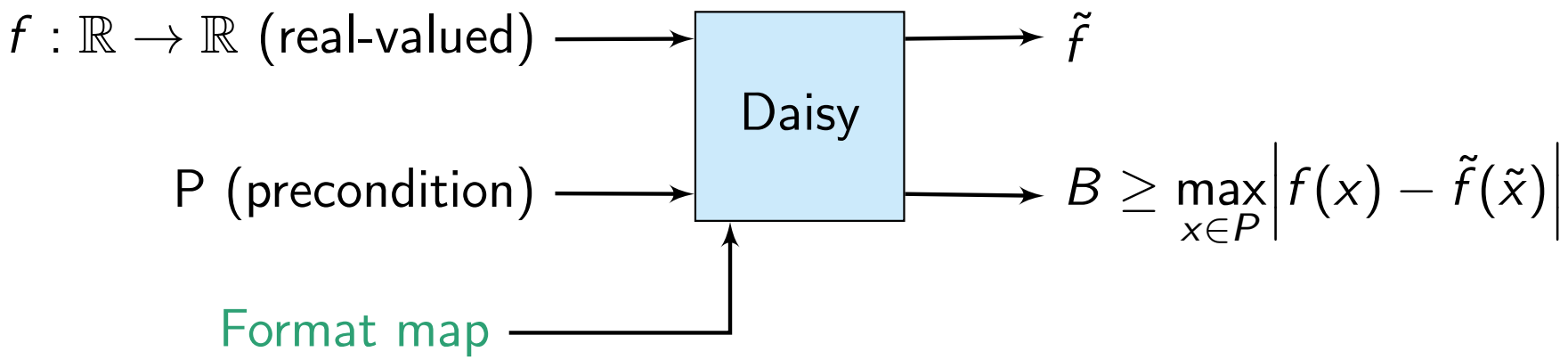

Technical details:

- Written in Scala
- Supports variable bindings (no control flow)
- Analysis using arbitrary-precision rationals

How Daisy works – example

```
def sine(a:Real) = {
  require(-1 <= a && a <= 1)
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sine = {
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```

```
abs-error: 2.507253663945146e-16,
real range: [-1.175, 1.175]
```

How Daisy works – Analysis

Two-step analysis:

- Real range analysis `real range: [-1.175, 1.175]`
- Roundoff error analysis `abs-error: 2.51e-16`

Range analysis using Interval Arithmetic

Interval arithmetic

$$[a; b] + [c; d] = [a + c; b + d]$$

$$[a; b] \times [c; d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

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Soundness of interval arithmetic

For $\circ \in \{+, -, \times, /\}$, $\circ$ should **over-approximate** $\circ$, i.e:

$$x \in [a; b] \wedge y[c; d] \implies x \circ y \in [a; b] \circ [c; d]$$

Floating-point numbers (Normalized and normal)

$$1.m_1 m_2 \ldots m_{p-1} \cdot \pm 2^e \qquad e_{min} \leq e \leq e_{max}$$

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Round-to-nearest, ties to even: $RN(x)$.

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$1 + \delta$ **abstraction**

$$RN(x) = x \cdot (1 + \delta) \qquad |\delta| \leq 2^{-p}$$

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So: $x \circ_{FP} y = (x \circ y) \cdot (1 + \delta)$

Floating-point arithmetic – Mixed-precision

Format	Precision (p)	e_{min}	e_{max}
32-bits	24	-126	+127
64-bits	53	-1022	+1023
128-bits	113	-16382	+16382

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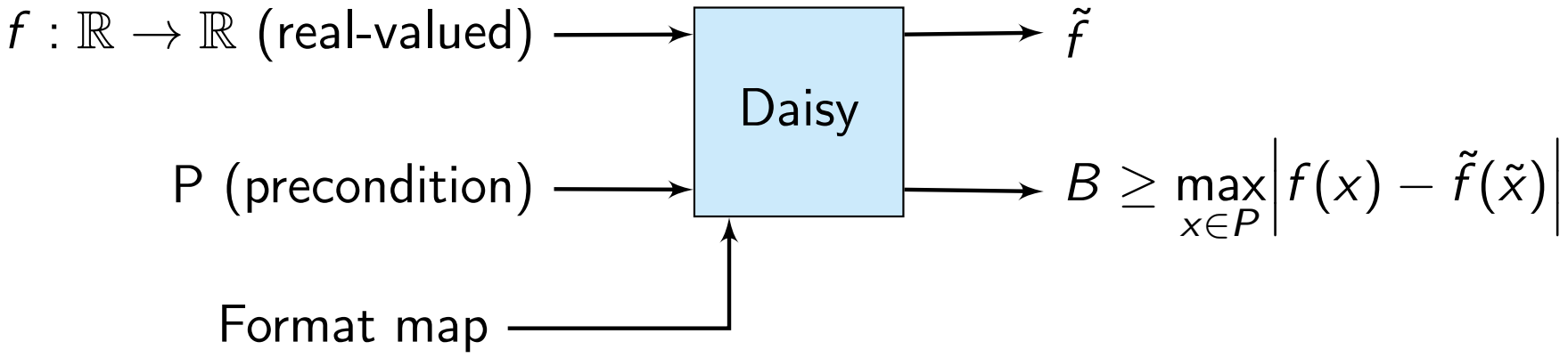
$1 + \delta$ abstraction for `Downcast`

$$\texttt{Downcast}(32, x_{64}) = x_{64} \cdot (1 + \delta) \qquad |\delta| \leq 2^{-p_{32}} = 2^{-24}$$

Formal verification of Daisy

Overview of certificate checking



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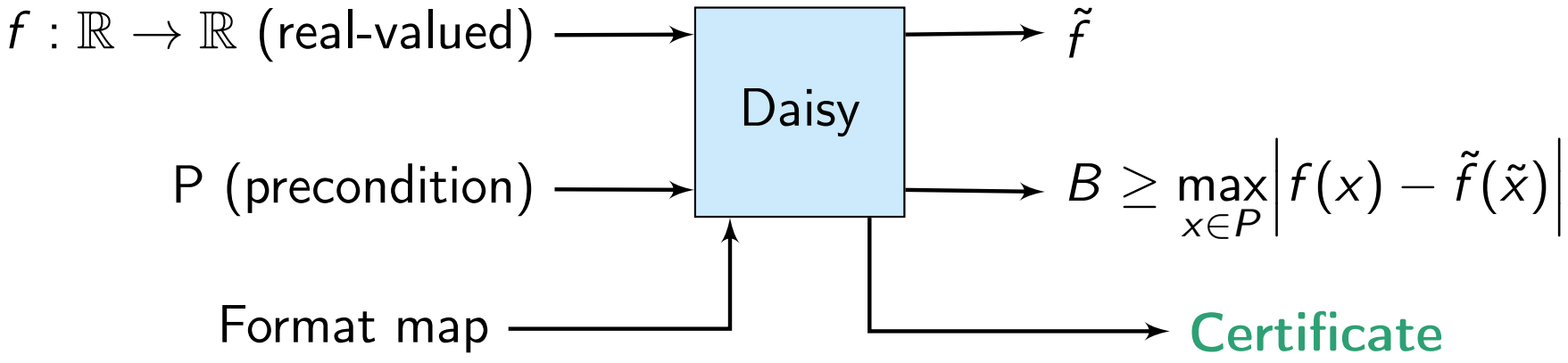

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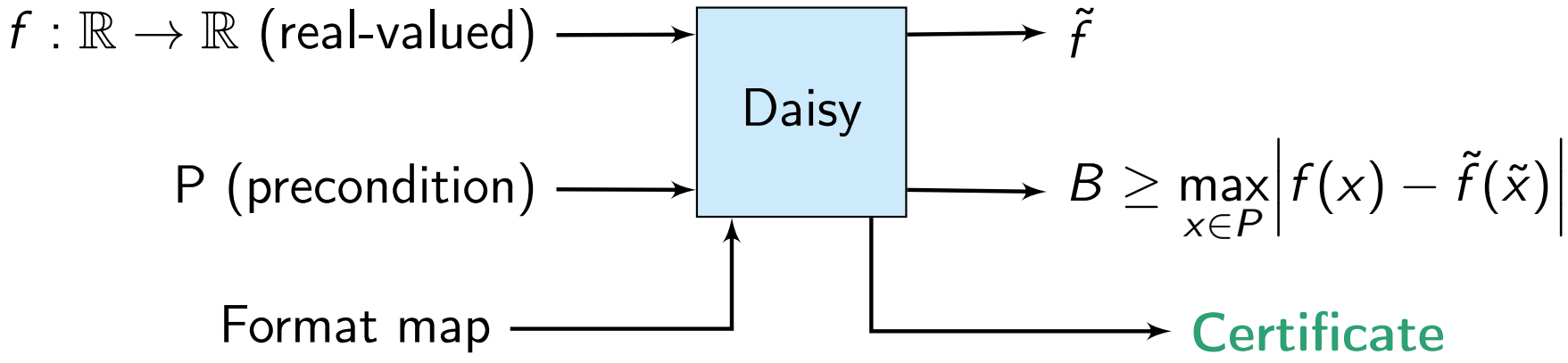
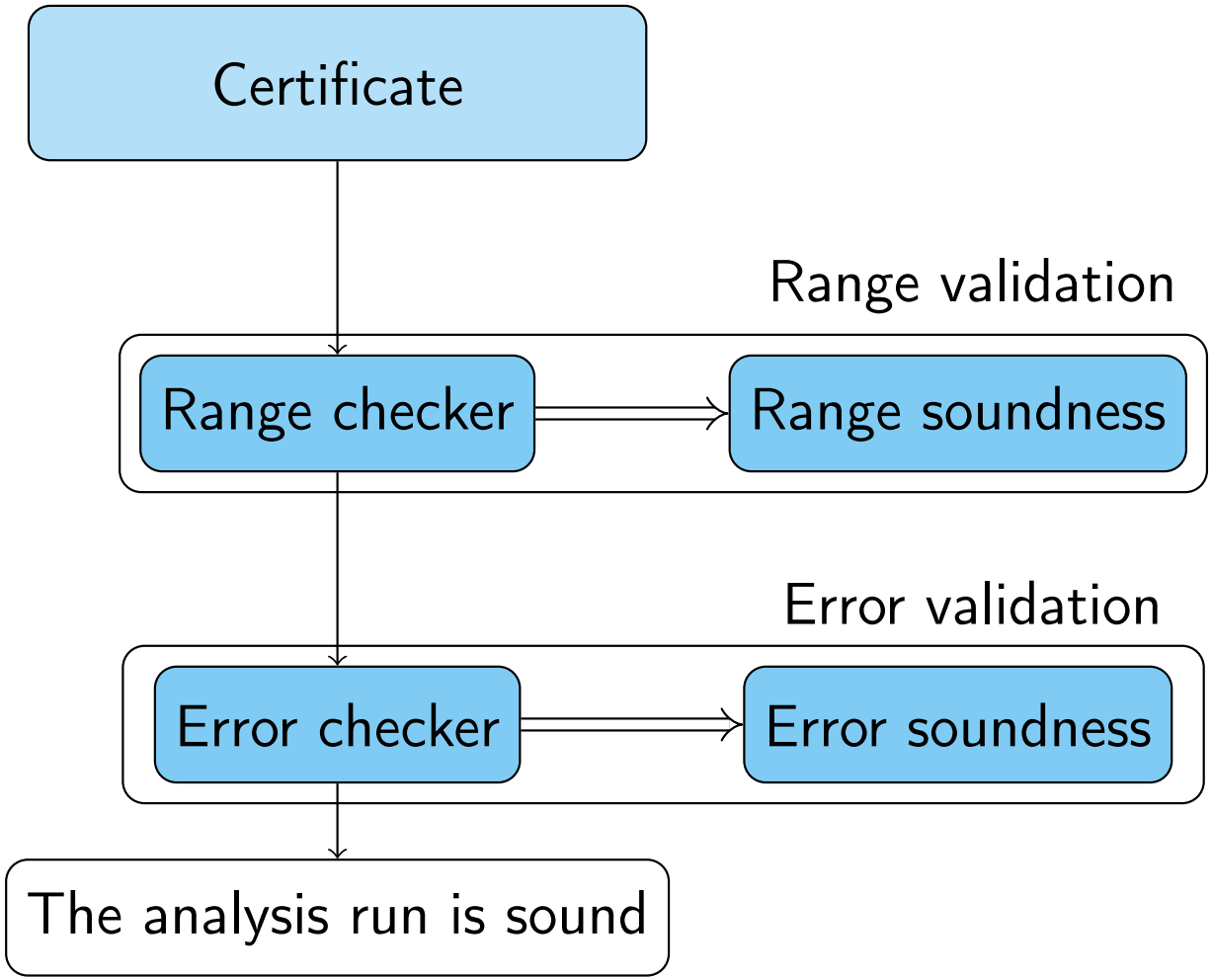


Certificate checker:

- computes its own roundoff error C
- checks that $B \geq C$

Formal soundness proof: $C \geq \max_{x \in P} \left| f(x) - \tilde{f}(\tilde{x}) \right|$

Overview of certificate checking – uniform precision



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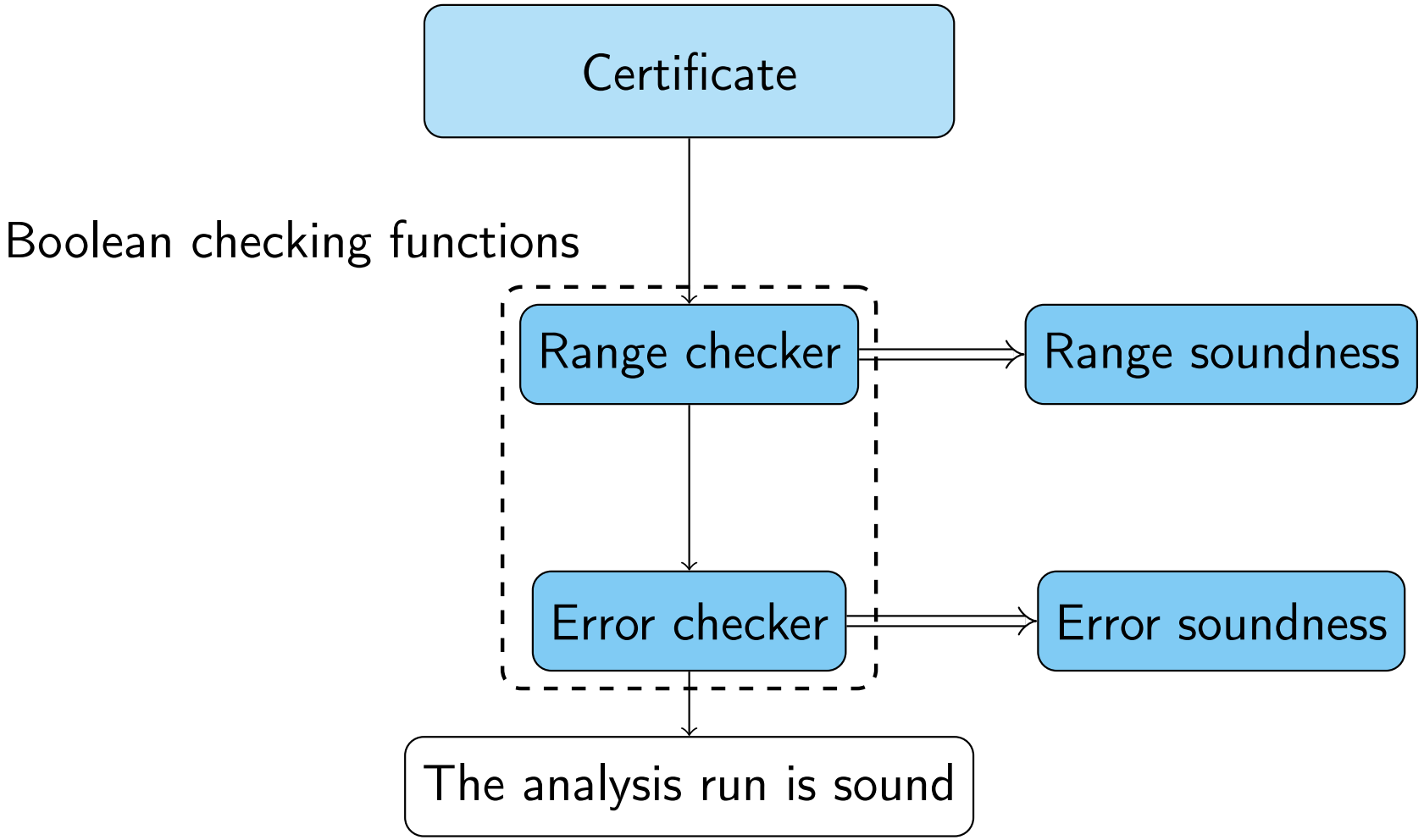

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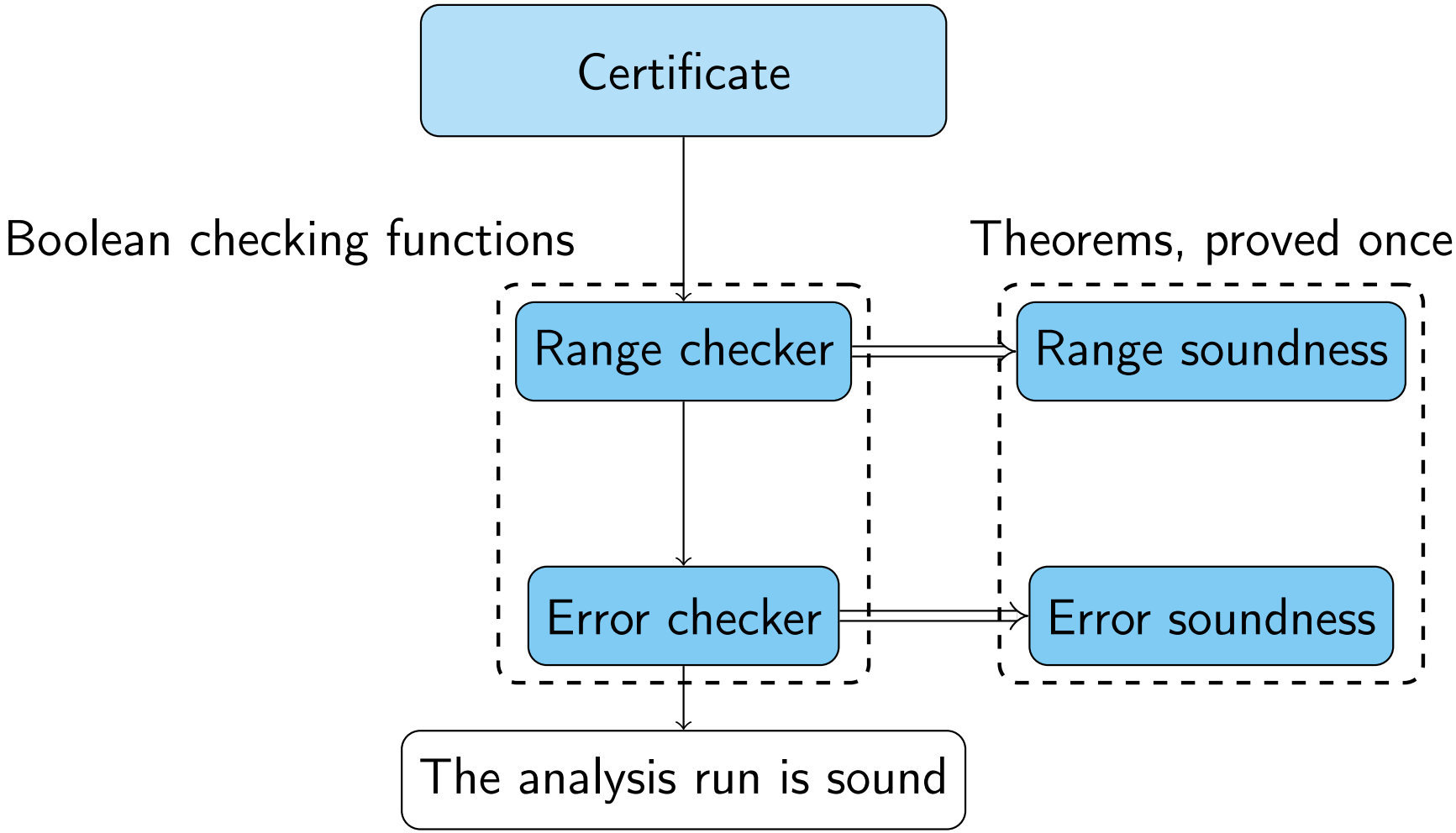
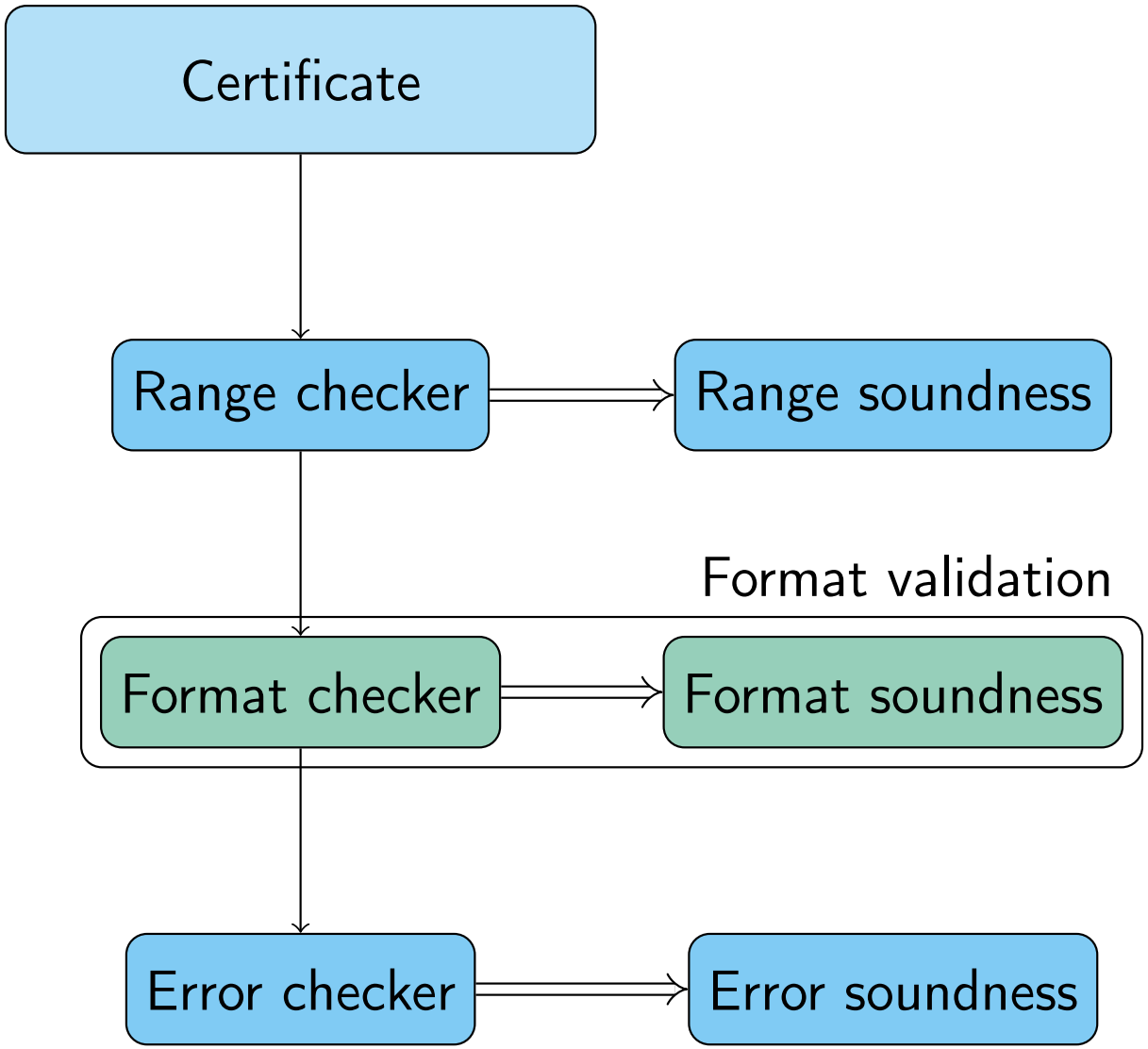


Adding support for mixed-precision



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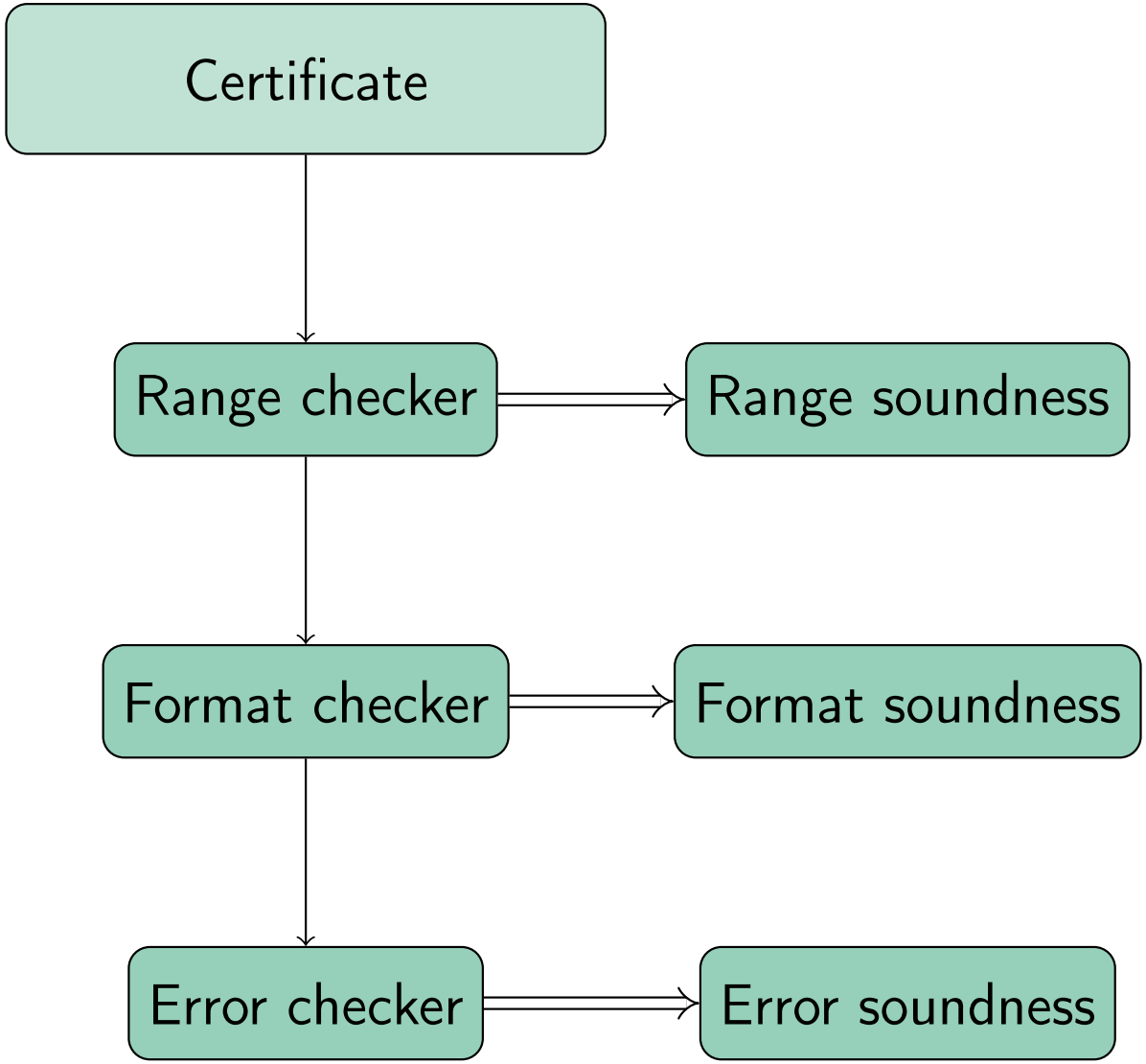

Certificate

What is encoded in a certificate of a function f?

- Definition of the expressions of f
- Precondition of f
- Analysis result (range and error) for each expression of f
- Format map for the variables of f

Certificate

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Definition ExpVara0 :exp Q := Var Q 0.
Definition UMinExpVara0 :exp Q := Unop Neg ExpVara0.
Definition MultUMinExpVara0ExpVara0 :exp Q := Binop Mult UMinExpVara0 ExpVara0.

Definition defVars_sine :( nat → option mType) := fun n ⇒
  if n =? 0 then Some M64 else if n =? 3 then Some M128
  else if n =? 4 then Some M64 else None.

Definition thePrecondition_sine : precond := fun (n: nat) ⇒
if n =? 0 then ( (−1)/(1), (1)/(1)) else (0/1,0/1).

Definition absenv_sine :analysisResult :=
fun (e: exp Q) ⇒
if (expEqBool e ExpVara0) then (((−1)/(1), (1)/(1)), (1)/(9007199254740992))
else if (expEqBool e UMinExpVara0 ) then (((−1)/(1), (1)/(1)), (1)/(9007199254740992))
else if (expEqBool e MultUMinExpVara0ExpVara0 ) then
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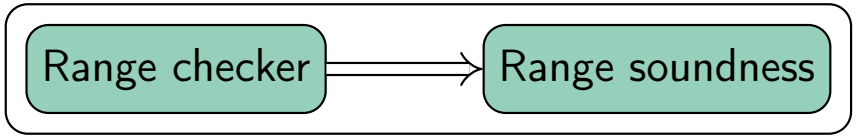
Definition thePrecondition_sine : precond := fun (n: nat) ⇒
if n =? 0 then ( (−1)/(1), (1)/(1)) else (0/1,0/1).
```

Analysis result

```
Definition absenv_sine : analysisResult :=
fun (e: exp Q) ⇒
if (expEqBool e ExpVara0) then (((−1)/(1), (1)/(1)), (1)/(9007199254740992))
else if (expEqBool e UMinExpVara0) then (((−1)/(1), (1)/(1)), (1)/(9007199254740992))
else if (expEqBool e MultUMinExpVara0ExpVara0 ) then
      (((−1)/(1), (1)/(1)),(24338891524382007210896477965516 9)/
            (730750818665451459101842416358141509827966271488))
```

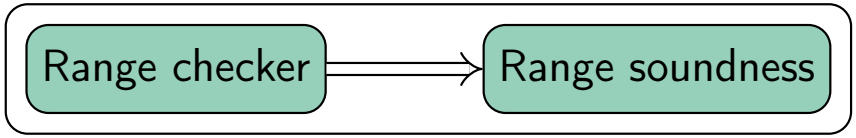
Range validation



Extension: case of `Downcast`

Check that the range of `Downcast m e` is equal to the range of `e`.

Range validation



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Soundness theorem

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Theorem validIntervalbounds_sound  e absenv P E:
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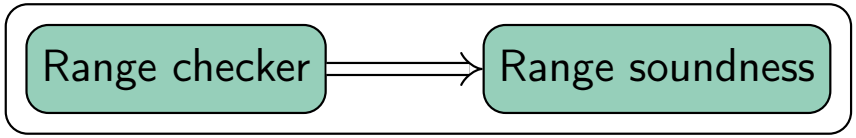

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Range check ok

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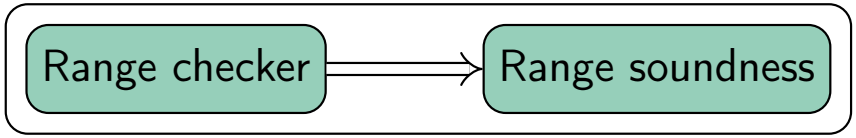

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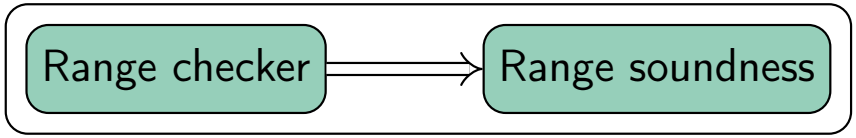
Range check ok

The real value of e is vR

Range validation



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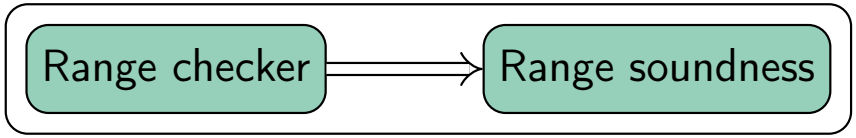
Range computed by Daisy

Range validation



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```

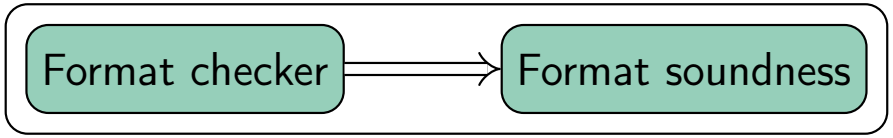
Range check ok

The real value of e is vR

Range computed by Daisy

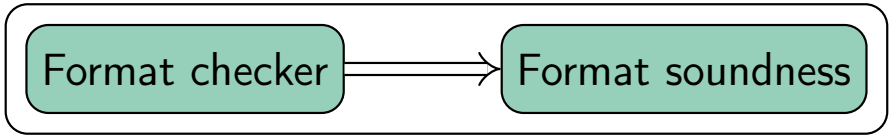
$\Longrightarrow$ The real range analysis is sound for e.

Format validation



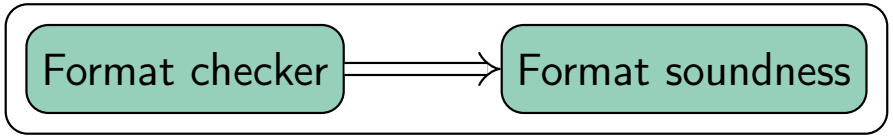
- Completely new validation phase

Format validation



- Completely new validation phase
- Checks that the precision in the expressions are coherent:

Format validation



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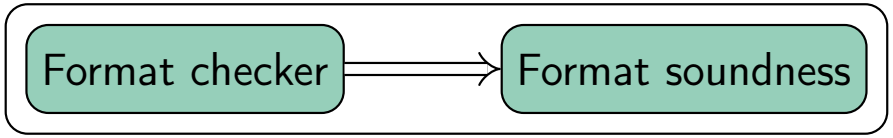
Format validation



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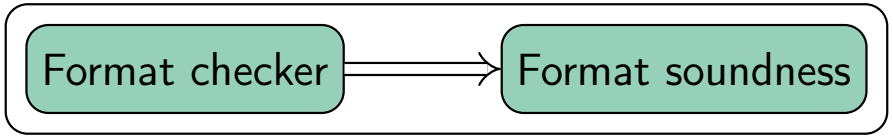
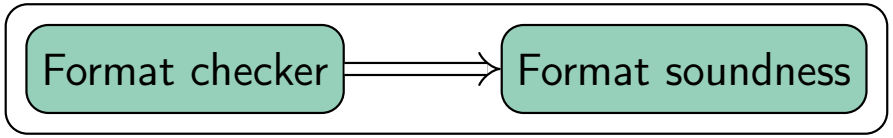


- Completely new validation phase
- Checks that the precision in the expressions are coherent:
  - `(x:Double) + (y:Float)` should be `Double`
  - $\mathit{Downcast}(64, x_{32})$ is incorrect
- Difficult to design: I wanted to have a stronger property

Format validation

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Theorem formatSoundnessExp  e defVars E v m Gamma:
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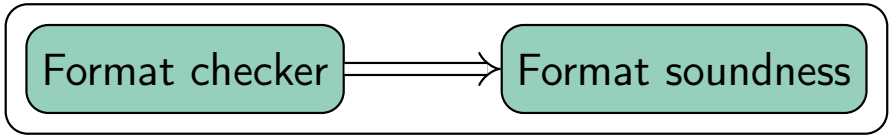
Format validation

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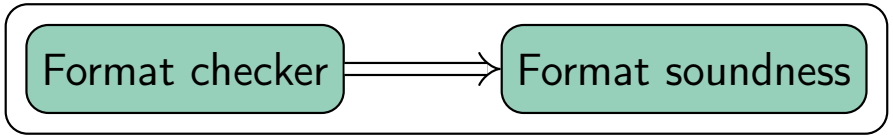
Format check of Γ ok

Format validation

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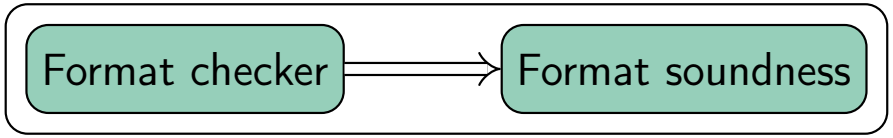
Format check of Γ ok

In Scala, e has type m

Format validation

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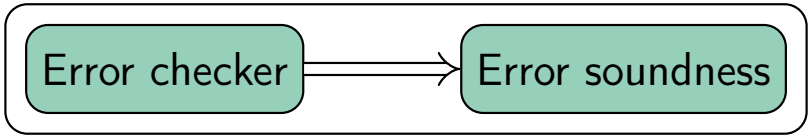
Format check of Γ ok

In Scala, e has type m

$\Longrightarrow$ The types used in the analysis are correct with respect to the Scala semantics.

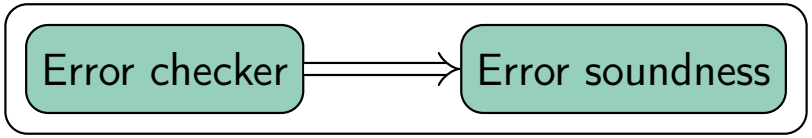
Error validation



Case of `Downcast`

Example: `val y: Float = (x: Double). toFloat`

Error validation

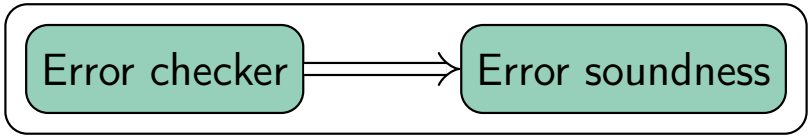


Case of `Downcast`

Example: `val y: Float = (x: Double). toFloat` $\rightsquigarrow$ `Downcast 32` $\tilde{x}_{64}$.

$$|x - (\texttt{Downcast 32 } \tilde{x}_{64})| \leq |x - \tilde{x}_{64}| +$$

Error validation



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$$|x - (\texttt{Downcast 32 } \tilde{x}_{64})| \leq |x - \tilde{x}_{64}| + |\tilde{x}_{64} - (\texttt{Downcast 32 } \tilde{x}_{64})|$$

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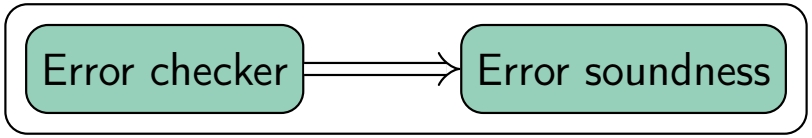

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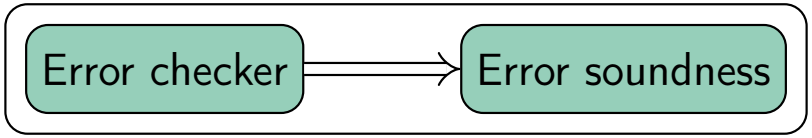
$$\begin{aligned} |x - (\texttt{Downcast 32 } \tilde{x}_{64})| &\leq |x - \tilde{x}_{64}| + |\tilde{x}_{64} - (\texttt{Downcast 32 } \tilde{x}_{64})| \\ &\leq |x - \tilde{x}_{64}| + |\tilde{x}_{64}| \cdot 2^{-24} \end{aligned}$$

Error validation

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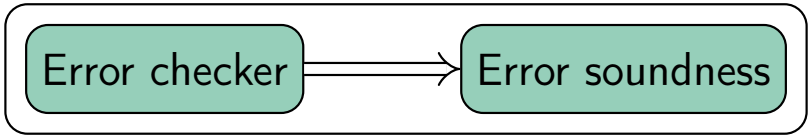
  eval_exp E1 (toREval e) vR M0 →
  eval_exp E2 e vF m →
  |vR − vF| <= snd (absenv e).
```

Range check ok

Error validation

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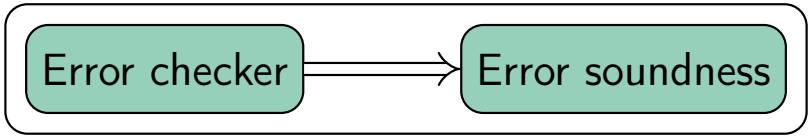


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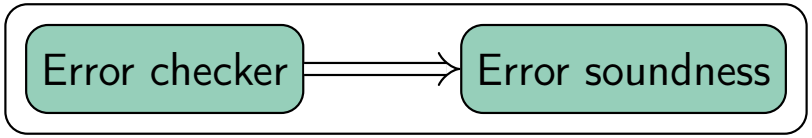


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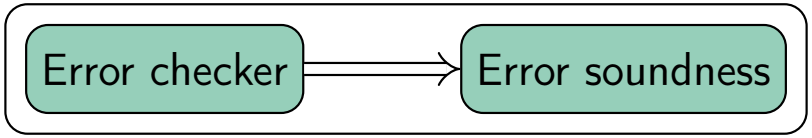


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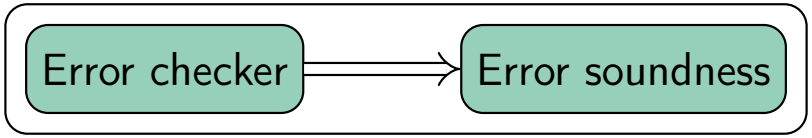


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Range check ok

Format check ok

Error check ok

Using reals, e evaluates to vR

In Scala, e evaluates to vF having type m

$\Longrightarrow$ The result of the analysis is an upper-bound on the roundoff error, so the run roundoff error analysis is sound.

Related Work

Related Work

Tool	Fluctuat[1]	Daisy	FPTaylor[2]	Gappa[3]
Formal verification	✗	✓	✓	✓
Mixed-precision	✓	✓	✗	✗
Control-flow	✓	✗	✗	✗
Variable-bindings	✓	✓	✗	✓
Automation	Full	Full	Full	Partial
Provers	N/A	Coq, HOL4	HOL-Light	Coq
Domain	Custom	IA	N/A	IA

[1] Goubault and Putot. "Static Analysis of Numerical Algorithms". *SAS 2006*.

[2] Solovyev et al. "Rigorous Estimation of Floating-Point Round-off Errors with Symbolic Taylor Expansions". *FM 2015*.

[3] Boldo, Filliâtre, and Melquiond. "Combining Coq and Gappa for Certifying Floating-Point Programs". *ICM, CICM 2009*.

Conclusion

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I extended the formal validation of Daisy to mixed-precision.

Daisy is the first formally verified tool supporting mixed-precision.

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Daisy is the first formally verified tool supporting mixed-precision.

Also:

- Formally implemented a part of Affine Arithmetic
- Helped create an automata demo for the "Girl's Day"
- Designed and conducted a practical session on Interproc, for a Master's course on Static Analysis
- Regular meetings within the PLV group

Code statistics

- Daisy: 11,000 lines of Scala code
- Uniform-precision development: 5,000 (Coq), 6,000 (HOL4)
- Mixed-precision development: 6,500 (Coq), 7,100 (HOL4)

Girl's Day

Bahnhof Scheidt

Halberg

Flughafen Saarbrücken

Hauptbahnhof
Saarbrücken

Völklingen

Burbach Mitte

Universität des Saarlandes

Schloss Saarbrücken

Staatstheater